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A class of zero product determined Lie algebras

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ABSTRACT

Let L be a Lie algebra over a field F . We say that L is zero product determined if, for every F -linear space V and every bilinear map $\varphi : L \times L \rightarrow V$, the following condition holds. If $\varphi(x, y) = 0$ whenever $[x, y] = 0$, then there exists a linear map f from $[L, L]$ to V such that $\varphi(x, y) = f([x, y])$ for all $x, y \in L$. This article shows that every parabolic subalgebra \mathfrak{p} of a (finite-dimensional) simple Lie algebra defined over an algebraically closed field is always zero product determined. Applying this result, we present a method different from that of Wang et al. (2010) [9] to determine zero product derivations of \mathfrak{p} , and we obtain a definitive solution for the problem of describing two-sided commutativity-preserving maps on \mathfrak{p} .

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1. Introduction

The concept of zero product determined Lie (resp., associative, Jordan) algebras was recently introduced by [3] and further studied by [5]. The original motivation for introducing these concepts emerges from the discovery that certain problems concerning linear maps on algebras, such as describing linear maps preserving commutativity or zero products, can be effectively treated by first examining bilinear maps satisfying certain related conditions [4]. Let us recall the following definition. Let F be a (fixed) field, and let A be an algebra over F . Let A^2 denote the F -linear span of all elements of the form xy for $x, y \in A$. The algebra A is called *zero product determined* if for every linear space X over F and every bilinear map $\{\cdot, \cdot\} : A \times A \rightarrow X$, the following holds. If $\{x, y\} = 0$ whenever

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$xy = 0$, then there exists a linear map $f : A^2 \rightarrow X$ such that $\{x, y\} = f(xy)$ for all $x, y \in A$. If the ordinary product is replaced by the Lie product, then it is said that A is zero Lie product determined. It should be noted that the problem of studying zero product determined algebras is nontrivial in the sense that there do exist associative and Lie algebras that are not zero product determined (see [5] for examples). The authors proved in [3] that the matrix algebra $M_n(R)$, $n \geq 2$, where R is any unital algebra, is always zero product determined. Moreover, if R is zero Lie product determined, then so is $M_n(R)$. In [5], M. Grašić showed that the Lie algebra of all $n \times n$ skew-symmetric matrices over an arbitrary field F of characteristic not 2 is zero product determined, as is the simple Lie algebra of the symplectic type over the above field F . The purpose of this paper is to extend the results from [5] to all parabolic subalgebras of the finite-dimensional simple Lie algebras over algebraically closed fields of characteristic 0. We also show the applicability of our main theorem to the study of zero product derivations and commutativity-preserving maps.

2. Basic Theorem

In this paper, the notation concerning Lie algebras mainly follows [6]. Let F be an algebraically closed field of characteristic 0. We denote by \mathfrak{g} a (finite-dimensional) simple Lie algebra over F of rank l . By \mathfrak{h} we denote a fixed Cartan subalgebra of \mathfrak{g} , and by Φ we denote the corresponding root system of \mathfrak{g} . Let Δ be a fixed base of Φ , and let Φ^+ (resp., Φ^-) be the set of positive (resp., negative) roots relative to Δ . The roots in Δ are called *simple*. For the base Δ of Φ , let $\mathfrak{d}_\Delta = \{d_\alpha \mid \alpha \in \Delta\}$ be the dual basis of \mathfrak{h} relative to Δ . Namely, $\beta(d_\alpha)$ takes the value 0 when $\beta \neq \alpha \in \Delta$, and it takes the value 1 when $\beta = \alpha \in \Delta$. Each root β can be written as $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}$. The integer $\sum_{\alpha \in \Delta} k_\alpha$ is called the *height* of β , which we denote by $ht \beta$. We denote by $\ker \alpha$, for $\alpha \in \Phi$, the kernel of α in \mathfrak{h} . Let $\mathfrak{g} = \mathfrak{h} + \sum_{\beta \in \Phi} \mathfrak{g}_\beta$ be the root space decomposition of \mathfrak{g} , where $\mathfrak{g}_\beta = \{x \in \mathfrak{g} \mid [h, x] = \beta(h)x, \forall h \in \mathfrak{h}\}$ is the root space relative to $\beta \in \Phi$. For each $\alpha \in \Phi^+$, let e_α be a non-zero element of \mathfrak{g}_α . Then, there is a unique element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ span a three-dimensional simple subalgebra of \mathfrak{g} isomorphic to $sl(2, F)$ via $e_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ forms the basis of \mathfrak{g} . If $\alpha, \beta, \alpha + \beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a non-zero scalar multiple of $e_{\alpha+\beta}$, because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Define $N_{\alpha, \beta}$ so that $[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta}$, which we call the *structure constants* of \mathfrak{g} . If $x, y \in \mathfrak{g}$, define $k(x, y) = \text{Tr}(\text{ad } x \cdot \text{ad } y)$. Then k is a symmetric bilinear form of \mathfrak{g} called the *Killing form* of \mathfrak{g} . It is well known that the restriction of the Killing form of \mathfrak{g} to \mathfrak{h} is non-degenerate. Thus for each $\phi \in \mathfrak{h}^*$ there exists a unique $t_\phi \in \mathfrak{h}$ such that $k(t_\phi, h) = \phi(h)$ for all $h \in \mathfrak{h}$, and the map from \mathfrak{h}^* to \mathfrak{h} , defined by $\phi \mapsto t_\phi$, is an isomorphic map. A symmetric bilinear form (\cdot, \cdot) is defined on the l -dimensional real vector space spanned by Φ , which is dual to the Killing form of \mathfrak{h} . For $\beta \in \Phi$, we know that h_β is a non-zero multiple of t_β . More definitely, $h_\beta = \frac{2t_\beta}{(\beta, \beta)}$. A subalgebra \mathfrak{p} of \mathfrak{g} is called *parabolic* if it includes some Borel subalgebra of \mathfrak{g} . For a given subset π of Δ , define \mathfrak{p} (relative to π) to be the subalgebra of \mathfrak{g} generated by all \mathfrak{g}_α , $\alpha \in \Delta \cup \{-\pi\}$ along with \mathfrak{h} . Let $\Phi_\pi = \mathbb{Z}\pi \cap \Phi$, $\Phi_\pi^+ = \Phi_\pi \cap \Phi^+$, $\Phi_\pi^- = \Phi_\pi \cap \Phi^-$. In fact, $\mathfrak{p} = \mathfrak{h} + \sum_{\alpha \in \Phi^+ \cup \Phi_\pi^-} \mathfrak{g}_\alpha$. It is evident that if $\pi = \Delta$, then \mathfrak{p} is \mathfrak{g} itself. If $\pi = \emptyset$, then \mathfrak{p} is a Borel subalgebra of \mathfrak{g} , which we denote by \mathfrak{b} . It is well known that every parabolic subalgebra of \mathfrak{g} is conjugate under an automorphism to some \mathfrak{p} . From this point of view, to determine a bilinear map on an arbitrary parabolic subalgebra, we only need to determine those maps on \mathfrak{p} . In the following, we always denote by \mathfrak{p} the parabolic subalgebra of \mathfrak{g} relative to a fixed subset π of Δ . We now present the Basic Theorem as follows.

Basic Theorem. *Let \mathfrak{p} be a parabolic subalgebra of \mathfrak{g} relative to a subset π of Δ . Then, \mathfrak{p} is zero product determined. More concretely, if a bilinear map φ from $\mathfrak{p} \times \mathfrak{p}$ to a linear space V over F satisfies the property that $\varphi(x, y) = 0$ whenever $\{x, y\} = 0$, then there exists a linear map f from $[\mathfrak{p}, \mathfrak{p}]$ to V such that $\varphi(x, y) = f([x, y])$ for all $x, y \in \mathfrak{p}$.*

One should note first that the bilinear map φ mentioned in the Basic Theorem is skew symmetric. In fact, by $\varphi(x + y, x + y) = \varphi(x, x) = \varphi(y, y) = 0$, $\varphi(x, y) = -\varphi(y, x)$ for all $x, y \in \mathfrak{p}$. Because the set $\{h_\alpha, e_\beta \mid \alpha \in \pi, \beta \in \Phi^+ \cup \Phi_\pi^-\}$ forms the basis of $[\mathfrak{p}, \mathfrak{p}]$, to define a linear map f from $[\mathfrak{p}, \mathfrak{p}]$ to V , we only need to define its action on h_α for $\alpha \in \pi$ as well as on e_β for $\beta \in \Phi^+ \cup \Phi_\pi^-$ and then extend it

linearly. We now use φ to induce a linear map f from $[\mathfrak{p}, \mathfrak{p}]$ to V by defining the action of f on the basis of $[\mathfrak{p}, \mathfrak{p}]$ as follows.

- $f(h_\alpha) = \varphi(e_\alpha, e_{-\alpha})$ for $\alpha \in \pi$;
- For each $\beta \in \Phi^+ \cup \Phi_\pi^-$, we choose $d_\beta \in \mathfrak{h}$ (depending on β) such that $\beta(d_\beta) = 1$, and we define the action of f on e_β as $f(e_\beta) = \varphi(d_\beta, e_\beta)$.

Lemma 2.1. $\varphi(h, e_\beta) = f([h, e_\beta])$ for all $h \in \mathfrak{h}$ and $\beta \in \Phi^+ \cup \Phi_\pi^-$.

Proof. By $[h - \beta(h)d_\beta, e_\beta] = 0$, then $\varphi(h - \beta(h)d_\beta, e_\beta) = 0$, which immediately implies that $\varphi(h, e_\beta) = f([h, e_\beta])$. \square

Lemma 2.2. For $\alpha, \gamma \in \Phi^+ \cup \Phi_\pi^-$, if $\alpha + \gamma \neq 0$, then $\varphi(e_\alpha, e_\gamma) = f([e_\alpha, e_\gamma])$.

Proof. If $\alpha + \gamma$ is not a root, then $[e_\alpha, e_\gamma] = 0$, and thus, the assertion obviously holds. Suppose that $\alpha + \gamma$ is a root β . Then, either $\beta + \alpha$ or $\beta + \gamma$ fails to be a root. Assume, without loss of generality, that $\beta + \alpha \notin \Phi$. Then, $[e_\beta, e_\alpha] = 0$. Choose $h \in \mathfrak{h}$ such that $\gamma(h) = 0$ and $\beta(h) = -N_{\alpha, \gamma}$. Then, $[h + e_\alpha, e_\beta + e_\gamma] = 0$, which implies that

$$\varphi(h + e_\alpha, e_\beta + e_\gamma) = 0.$$

Applying Lemma 2.1, we note that

$$\begin{aligned} \varphi(e_\alpha, e_\gamma) &= -\varphi(h, e_\beta) = -f([h, e_\beta]) \\ &= -\beta(h)f(e_\beta) = N_{\alpha, \gamma}f(e_\beta) \\ &= f([e_\alpha, e_\gamma]). \quad \square \end{aligned}$$

Lemma 2.3. Let $\beta \in \Phi_\pi^+$. Then, $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$ if there exist two distinct roots $\gamma, \alpha \in \Phi_\pi^+$ such that β, γ, α satisfy the following three conditions.

- The set $\{\beta, \gamma, \alpha\}$ is linearly dependant;
- $\beta + \alpha, \beta + \gamma, \gamma - \alpha$ all are not roots;
- $\varphi(e_\gamma, e_{-\gamma}) = f([e_\gamma, e_{-\gamma}])$ and $\varphi(e_\alpha, e_{-\alpha}) = f([e_\alpha, e_{-\alpha}])$.

Proof. By (i), we may assume that $\beta = a\alpha + b\gamma$ with $a, b \in F$. Because the map from \mathfrak{h}^* to \mathfrak{h} , as defined by $\phi \mapsto t_\phi$, is an isomorphic map, we note that $t_\beta = at_\alpha + bt_\gamma$. Recalling that $h_\beta = \frac{2t_\beta}{(\beta, \beta)}$, $h_\beta = a_1h_\alpha + b_1h_\gamma$, where $a_1 = \frac{(\alpha, \alpha)}{(\beta, \beta)}a$, $b_1 = \frac{(\gamma, \gamma)}{(\beta, \beta)}b$. By (ii), we have that $[e_\beta, e_\alpha] = [e_{-\beta}, e_{-\alpha}] = 0$, $[e_\beta, e_\gamma] = [e_{-\beta}, e_{-\gamma}] = 0$, and $[e_\gamma, e_{-\alpha}] = [e_{-\gamma}, e_\alpha] = 0$. Thus, one can easily see that

$$[e_\beta + a_1e_{-\alpha} + b_1e_{-\gamma}, e_{-\beta} + e_\alpha + e_\gamma] = 0,$$

from which we note that

$$\varphi(e_\beta + a_1e_{-\alpha} + b_1e_{-\gamma}, e_{-\beta} + e_\alpha + e_\gamma) = 0.$$

Applying condition (iii) of this lemma, we note that

$$\begin{aligned}
\varphi(e_\beta, e_{-\beta}) &= a_1\varphi(e_\alpha, e_{-\alpha}) + b_1\varphi(e_\gamma, e_{-\gamma}) \\
&= f(a_1h_\alpha + b_1h_\gamma) = f(h_\beta) \\
&= f([e_\beta, e_{-\beta}]). \quad \square
\end{aligned}$$

Lemma 2.4. $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$ for all $\beta \in \Phi_\pi$.

Proof. Because φ is skew symmetric, we need only prove the result for the case that $\beta \in \Phi_\pi^+$. The proof is divided into three parts for Φ of different types.

Case 1. Φ is type G_2 .

In this case, we arrange the basis of the root system as $\Delta = \{\alpha_1, \alpha_2\}$, where α_1 is a long root and α_2 is a short root. If π is the empty set or π has only one element, then the result obviously holds. Now, we consider the case that $\pi = \Delta$. Then

$$\Phi_\pi^+ = \Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}.$$

- If β is α_1 or α_2 , then the definition of f implies that $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$.
- If β is the maximal root $2\alpha_1 + 3\alpha_2$, choose α to be α_1 and choose γ to be α_2 . Then, β , α , and γ satisfy the three conditions of Lemma 2.3. Thus, $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$.
- If β is the root $\alpha_1 + 3\alpha_2$, set α to be α_2 and set γ to be $2\alpha_1 + 3\alpha_2$. Then β , α and γ satisfy the three conditions of Lemma 2.3. We also have that $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$.
- Either β takes the root $\alpha_1 + 2\alpha_2$, or it takes the root $\alpha_1 + \alpha_2$. We choose α to be α_1 and choose γ to be $\alpha_1 + 3\alpha_2$. Then, β , α and γ satisfy the three conditions of Lemma 2.3. The assertion holds.

So $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$ for all $\beta \in \Phi_\pi^+$.

Case 2. All roots in Φ have the same length.

In this case, we provide a proof by induction on $ht\beta$. If $ht\beta = 1$, then the assertion holds by definition of f . Suppose that the assertion holds for $\gamma \in \Phi_\pi^+$ with height k . Consider root $\beta \in \Phi_\pi^+$ with height $k + 1$. There exists some $\alpha \in \pi$ such that $\beta - \alpha \in \Phi_\pi^+$. Denote $\beta - \alpha$ by γ . Then $\varphi(e_\gamma, e_{-\gamma}) = f([e_\gamma, e_{-\gamma}])$, because of the induction assumption. Because all roots in Φ have the same length, we know that $\beta + \alpha$, $\beta + \gamma$, $\gamma - \alpha$ all are not roots. Thus, β , α and γ satisfy the conditions of Lemma 2.3, such that $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$.

Case 3. Φ has two root lengths and is not of type G_2 .

We again use induction on $ht\beta$ to prove that $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$ for $\beta \in \Phi_\pi^+$. If $ht\beta = 1$, then the assertion holds. Assume the assertion holds for $\gamma \in \Phi_\pi^+$ with height not larger than k . Consider root $\beta \in \Phi_\pi^+$ with height $k + 1$. There exists some $\alpha \in \pi$ such that $\beta - \alpha \in \Phi_\pi^+$. Denote $\beta - \alpha$ by γ . If $\gamma - \alpha$, denoted by γ_1 , is a root, then $\beta + \alpha$, $\beta + \gamma_1$ and $\gamma_1 - \alpha$ all are not roots. Then, β , α and γ_1 satisfy the conditions of Lemma 2.3. Thus, $\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}])$. Now suppose that $\gamma - \alpha$ is not a root. Note that $\beta + \alpha$ and $\beta + \gamma$ cannot both be roots. If $\beta + \alpha$ and $\beta + \gamma$ both are not roots, then β , α and γ satisfy the conditions of Lemma 2.3. Thus,

$$\varphi(e_\beta, e_{-\beta}) = f([e_\beta, e_{-\beta}]).$$

If $\beta + \alpha$ is a root but $\beta + \gamma$ is not a root, then one may verify that $\beta + \alpha$, α , γ satisfy the conditions

of Lemma 2.3, so

$$\varphi(e_{\beta+\alpha}, e_{-\beta-\alpha}) = f([e_{\beta+\alpha}, e_{-\beta-\alpha}]).$$

Moreover, one may verify that $\beta, \beta + \alpha$ and γ also satisfy the conditions of Lemma 2.3. Applying Lemma 2.3, we also note that

$$\varphi(e_{\beta}, e_{-\beta}) = f([e_{\beta}, e_{-\beta}]).$$

If $\beta + \gamma$ is a root and $\beta + \alpha$ is not a root, by an analogous process, we note that

$$\varphi(e_{\beta}, e_{-\beta}) = f([e_{\beta}, e_{-\beta}]). \quad \square$$

Combining Lemma 2.2 with Lemma 2.4, we note that

Lemma 2.5. $\varphi(e_{\beta}, e_{\gamma}) = f([e_{\beta}, e_{\gamma}])$ for all $\beta, \gamma \in \Phi^+ \cup \Phi_{\pi}^-$.

With Lemma 2.1 and Lemma 2.5, we are now ready to prove the Basic Theorem.

Proof of the Basic Theorem. Let f be defined as above. We now show that $\varphi(x, y) = f([x, y])$ for all $x, y \in \mathfrak{p}$. Express x and y as

$$x = h + \sum_{\beta \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} e_{\beta}, \quad y = d + \sum_{\gamma \in \Phi^+ \cup \Phi_{\pi}^-} b_{\gamma} e_{\gamma},$$

where $h, d \in \mathfrak{h}$ and $a_{\beta}, b_{\gamma} \in F$. Recalling that φ is skew symmetric and applying Lemma 2.1 and Lemma 2.5, we note that

$$\begin{aligned} \varphi(x, y) &= \varphi\left(h + \sum_{\beta \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} e_{\beta}, d + \sum_{\gamma \in \Phi^+ \cup \Phi_{\pi}^-} b_{\gamma} e_{\gamma}\right) \\ &= \varphi(h, d) + \sum_{\gamma \in \Phi^+ \cup \Phi_{\pi}^-} b_{\gamma} \varphi(h, e_{\gamma}) - \sum_{\beta \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} \varphi(d, e_{\beta}) + \sum_{\beta, \gamma \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} b_{\gamma} \varphi(e_{\beta}, e_{\gamma}) \\ &= f([h, d]) + \sum_{\gamma \in \Phi^+ \cup \Phi_{\pi}^-} b_{\gamma} f([h, e_{\gamma}]) - \sum_{\beta \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} f([d, e_{\beta}]) \\ &\quad + \sum_{\beta, \gamma \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} b_{\gamma} f([e_{\beta}, e_{\gamma}]) \\ &= f\left(\left[h + \sum_{\beta \in \Phi^+ \cup \Phi_{\pi}^-} a_{\beta} e_{\beta}, d + \sum_{\gamma \in \Phi^+ \cup \Phi_{\pi}^-} b_{\gamma} e_{\gamma}\right]\right) \\ &= f([x, y]). \quad \square \end{aligned}$$

3. Application to zero product derivations

Recently, some researchers have become interested in generalizing *derivation* of Lie algebras. Leger and Luks introduced the concept of quasi-derivation of Lie algebras in [7]. Let L be a Lie algebra. A linear map f on L is called a *quasi-derivation* of L if there exists a linear map f' on $[L, L]$ such that

$$[f(x), y] + [x, f(y)] = f'([x, y]), \quad \forall x, y \in L.$$

In [7], it was shown that

$$Q\text{ Der}(L) = \text{Der}(L) + C(L)$$

if L is generated by special weight spaces, where $Q\text{ Der}(L)$ denotes the set of all quasi-derivations of L , and $C(L)$ indicates the centroid of L . In particular, for a parabolic subalgebra \mathfrak{p} of a simple Lie algebra of characteristic 0, each quasi-derivation of \mathfrak{p} was shown to be the sum of an inner derivation and a scalar multiplication map on \mathfrak{p} if $\text{rank}(\mathfrak{g}) \geq 2$. A linear map f on a Lie algebra L is called a *zero product derivation* of L if $[x, y] = 0$ implies that $[f(x), y] + [x, f(y)] = 0$. It is easy to verify that a quasi-derivation is a zero product derivation of L . As such the concept of zero product derivation is slightly more general than that of quasi-derivation. The problem of describing zero product Lie derivations for certain rings was first described by [1] (see Theorem 4 in that paper). In [9] we studied zero product derivations for parabolic subalgebras of simple Lie algebras and obtained the following theorem using a method that mainly depends on direct calculation. To apply the Basic Theorem of this article, we now prove this result in a different way.

Theorem 3.1. (See [9].) *If $\text{rank}(\mathfrak{g}) = 1$, then every linear map on \mathfrak{p} is a zero product derivation of \mathfrak{p} . If $\text{rank}(\mathfrak{g}) \geq 2$, then a zero product derivation of \mathfrak{p} is simply the sum of an inner derivation and a scalar multiplication map.*

Proof. Let ψ be a zero product derivation of \mathfrak{p} . Using ψ , we define $\varphi : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ so that $\varphi(x, y) = [\psi(x), y] + [x, \psi(y)]$ for $x, y \in \mathfrak{p}$. Then note that φ is bilinear, and if $[x, y] = 0$ for $x, y \in \mathfrak{p}$, then $\varphi(x, y) = [\psi(x), y] + [x, \psi(y)] = 0$, recalling that ψ is a zero product derivation of \mathfrak{p} . Applying the Basic Theorem, we can find a linear map f from $[\mathfrak{p}, \mathfrak{p}]$ to \mathfrak{p} such that $\varphi(x, y) = [\psi(x), y] + [x, \psi(y)] = f([x, y])$ for all $x, y \in \mathfrak{p}$. This implies that ψ is exactly a quasi-derivation of \mathfrak{p} . Applying Corollary 4.13 in [7], we note that ψ is the sum of an inner derivation and a scalar multiplication map in the case that $\text{rank}(\mathfrak{g}) \geq 2$. In the case that $\text{rank}(\mathfrak{g}) = 1$, the result obviously holds. \square

4. Application to commutativity-preserving maps

Much attention has been paid to commutativity-preserving problems on associative F -algebras, particularly matrix algebras. The earliest paper on such problems dates back to 1976, when Watkins [10] studied commutativity-preserving maps on the full matrix algebra M_n over a field F . To review the rather long and rich history of commutativity preserver problems, the reader is referred to the historic remarks in the book [2] by Brešar, Chebotar and Martindale. For a Lie algebra L , we say that x commutes with y if $[x, y] = 0$. An invertible linear map ψ on L is called a *two-sided commutativity-preserving map* if $[\psi(x), \psi(y)] = 0 \Leftrightarrow [x, y] = 0$ for $x, y \in L$. An invertible linear map ϕ on L is called a *quasi-automorphism* of L if there exists an invertible linear map $\tilde{\phi}$ on $[L, L]$ such that $[\phi(x), \phi(y)] = \tilde{\phi}([x, y])$ for all $x, y \in L$. Surveying the literature, we find that in 1981 Wong [11] studied invertible linear maps on Lie algebras that preserve commutativity. However, he only studied such maps on simple Lie algebras of linear types. We extend Wong's result to parabolic subalgebras of simple Lie algebras. Our goal is to reduce two-sided commutativity-preserving maps on \mathfrak{p} to quasi-automorphisms on \mathfrak{p} . Obviously, a quasi-automorphism of a Lie algebra L must be a two-sided commutativity-preserving map on L . Using the Basic Theorem presented in this article, we prove in Theorem 4.2 that if L is taken to be the parabolic subalgebra \mathfrak{p} of \mathfrak{g} , then the converse statement

also holds. To provide a definite description of two-sided commutativity-preserving maps on \mathfrak{p} , we must first characterize quasi-automorphisms of \mathfrak{p} . We note that this work has been done in our other article [8]. The main result of this article is as follows.

Theorem 4.1. (See [8].) *Let \mathfrak{g} be a simple Lie algebra of rank l over an algebraically closed field F of characteristic 0, and let \mathfrak{p} be an arbitrary parabolic subalgebra of \mathfrak{g} .*

- (i) *If $l = 1$, then every invertible linear map on \mathfrak{p} is a quasi-automorphism;*
- (ii) *If $l \geq 2$, then every quasi-automorphism of \mathfrak{p} is a composition of an automorphism and a non-zero scalar multiplication map on \mathfrak{p} .*

Applying Theorem 4.1 and the Basic Theorem of the present article, we now describe definitely two-sided commutativity-preserving maps on \mathfrak{p} .

Theorem 4.2. *An invertible linear map on \mathfrak{p} is a two-sided commutativity-preserving map if and only if it is a quasi-automorphism of \mathfrak{p} . More precisely,*

- (i) *if $l = 1$, then every invertible linear map on \mathfrak{p} is a two-sided commutativity-preserving map;*
- (ii) *if $l \geq 2$, then every two-sided commutativity-preserving map on \mathfrak{p} is a composition of an automorphism and a non-zero scalar multiplication map on \mathfrak{p} .*

Proof. A quasi-automorphism of \mathfrak{p} is a two-sided commutativity-preserving map on \mathfrak{p} . Let ψ be a two-sided commutativity-preserving map on \mathfrak{p} . ψ^{-1} is also such a map. Using ψ , we define $\varphi : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ so that $\varphi(x, y) = [\psi(x), \psi(y)]$ for all $x, y \in \mathfrak{p}$. Note that φ is bilinear, and if $[x, y] = 0$ for $x, y \in \mathfrak{p}$, then $\varphi(x, y) = [\psi(x), \psi(y)] = 0$. Applying the Basic Theorem, we find a linear map f from $[\mathfrak{p}, \mathfrak{p}]$ to \mathfrak{p} such that $\varphi(x, y) = [\psi(x), \psi(y)] = f([x, y])$ for all $x, y \in \mathfrak{p}$. Similarly, there exists a linear map f_1 from $[\mathfrak{p}, \mathfrak{p}]$ to \mathfrak{p} such that $[\psi^{-1}(x), \psi^{-1}(y)] = f_1([x, y])$ for all $x, y \in \mathfrak{p}$. Thus, f is invertible, with f_1 as its inverse. Therefore, ψ is precisely a quasi-automorphism of \mathfrak{p} . Applying Theorem 4.1, we obtain the definite description of ψ . \square

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